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UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél. (1) 39 63 55 11

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A NEW REPRESENTATION OF H^∞ SUBOPTIMAL CONTROLLERS

Pascal GAHINET

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A New Representation of H_∞ Suboptimal Controllers

Pascal M. Gahinet

INRIA
Domaine de Voluceau
Rocquencourt - BP 105
78153 Le Chesnay Cedex
France

Abstract: Elementary state-space concepts are used to derive a transparent solution to the stationary H_∞ control problem. The main contribution of this solution is a novel representation of suboptimal controllers in terms of a pair of parametric Riccati equations with a coupling constraint. Unlike the classical parametrization in terms of linear fractional transformation, this state-space representation has a homogeneous Riccati formulation which should help to make the most out of the suboptimal controller diversity. Potential applications include the design of reduced-order controllers and more generally the selection of suboptimal controllers to meet or optimize additional constraints.

Une Nouvelle Representation des Contrôleurs H_∞ Sous-Optimaux

Résumé: Une solution complète du problème H_∞ stationnaire est obtenue par des manipulations algébriques élémentaires. Ici l'outil central est le Lemme Réel Borné qui intervient déjà dans le calcul de la norme L_∞ d'une fonction de transfert. La principale contribution est de fournir une représentation de l'ensemble des contrôleurs sous-optimaux en terme d'équations de Riccati paramétrées. Cette représentation algébrique est plus maniable que la Q -paramétrisation et offre de nouvelles perspectives pour la synthèse de contrôleurs sous-optimaux. En particulier, la réduction de l'ordre et la prise en compte d'autres spécifications comme les marges de stabilité semblent des applications prometteuses.

1 Introduction

A wide range of significant control problems fall into the framework of H_∞ design. Well-known examples include disturbance attenuation, mixed-sensitivity design, model matching, and robust stability problems. Because of its natural formulation in the frequency domain, the H_∞ control problem was long regarded as beyond the reach of state-space techniques. Gradually however, it became apparent that state-space formulations actually lead to more concise and numerically appealing solutions. A cornerstone contribution to this growing awareness is of course the paper by Doyle *et al.* [5] where solvability of suboptimal problems is characterized in terms of two game Riccati equations (GRE). In addition, explicit formulas are given for a particular solution called “central” controller and a mathematically elegant parametrization of all suboptimal controllers is derived. Finally, these results are numerically appealing since computations essentially reduce to solving the well-known matrix Riccati equation. In the sequel, the acronym DGKF is used to refer to the authors of this paper.

Despite the simplicity of the results, the conceptual link between the original problem and the Riccati equations remains somewhat obscure in [5]. This connection has been further clarified with “purely state-space” derivations such as the linear quadratic game approach [11, 2] for the time-varying case, or the algebraic approach of [12] for the stationary case (usual H_∞ problem). In the latter paper in particular, the Bounded Real Lemma was first recognized as a simple device to turn the H_∞ problem into Riccati equations. This technique has been further used in [14]. Yet, it is our opinion that its transparency and implications have not been fully exposed, mostly because so far the main concern has revolved around reproducing DGKF’s results.

In the present paper, we take a different perspective and explore which type of suboptimal controller representation is obtained via the Bounded Real Lemma. A two-fold contribution emerges from this perspective. First, a straightforward solution to the H_∞ problem is proposed which clearly exposes the connection between the H_∞ constraint and the GRE’s. Then, a new representation is obtained for suboptimal controllers of order less than or equal to the plant order. In this representation, each controller is described by two coupled GRE’s which involve some of its state-space parameters. The usual pair of H_∞ Riccati equations is recovered as a special case corresponding to the central controller. The characteristics and potential of this new representation are explored in details. In particular, a comparison with DGKF’s parametrization of all suboptimal controllers reveals the complementarity of the two descriptions. Due to its homogeneous Riccati formulation, our representation seems well adapted to analytical and numerical manipulations. It also offers a convenient state-space framework for both reduced-order design and selection of the controller consistently with additional constraints or objectives.

The paper is organized as follows. In Section 2, the H_∞ control problem setup is recalled as well as the Bounded Real Lemma. Section 3 sketches the connection between this lemma and the classical results. Section 4 presents the main result which is a representation of suboptimal controllers in terms of parametric GRE’s. In Section 5, the usual pair of H_∞ Riccati equations is characterized as an extremal member of this parametric family, a fact which readily yields DGKF’s solvability conditions for suboptimal problems. Section 6 compares the new representation with DGKF’s Q -parametrization of all suboptimal controllers. Finally, Section 7 gives an illustrative example while section 8 discusses effective and potential applications to controller gain reduction, reduced-order controller design, and internal stability improvement.

2 Problem Setup

As usual in H_∞ control problems, consider a plant $G(s)$ which maps exogenous inputs w and control inputs u to controlled outputs z and measured outputs y . That is,

$$\begin{pmatrix} z \\ y \end{pmatrix} = G(s) \begin{pmatrix} w \\ u \end{pmatrix} \text{ with}$$

$$G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (sI - A)^{-1} \begin{pmatrix} B_1 & B_2 \end{pmatrix}. \quad (2.1)$$

Here $A \in \mathbb{R}^{n \times n}$ and z, y, w , and u are vectors of size p_1, p_2, m_1 , and m_2 , respectively, with the assumption that $m_1 \geq p_2$ and $p_1 \geq m_2$. When the plant G is closed by the output feedback law $u = K(s)y$, the closed-loop transfer function from w to z is given by the linear fractional transformation (LFT):

$$\mathcal{F}(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}. \quad (2.2)$$

The H_∞ suboptimal control problem of parameter γ consists of finding a dynamic real-rational controller $K(s)$ such that the feedback law $u = K(s)y$ internally stabilizes G and makes the closed-loop gain less than γ . That is:

$$\mathcal{P}_\infty(\gamma): \text{ find } K(s) \text{ internally stabilizing such that } \|\mathcal{F}(G, K)\|_\infty \leq \gamma \quad (2.3)$$

The optimal H_∞ attenuation γ_{opt} is then defined as the smallest $\gamma > 0$ for which the suboptimal problem $\mathcal{P}_\infty(\gamma)$ has a solution. Note that (2.3) allows for $\|\mathcal{F}(G, K)\|_\infty = \gamma$ so that the optimal case $\gamma = \gamma_{opt}$ will not require separate treatment.

Strictly speaking, solvability of $\mathcal{P}_\infty(\gamma)$ for γ large enough only requires that

(A1) (A, B_2, C_2) is stabilizable and detectable.

Although the present approach can be worked from this sole assumption, the complexity of the resulting formulas and calculations is detrimental to the clarity of the argument. To favor insight over technicalities, the discussion is therefore specialized to the Standard Problem (SP) by adding the following assumptions:

(A2) $D_{22} = D_{11} = 0$.

(A3) $D_{12}^T(D_{12}, C_1) = (I, 0)$ and $D_{21}(D_{21}^T, B_1^T) = (I, 0)$.

(A4) D_{12} has full column rank and D_{21} has full row rank.

(A5) (A, B_1, C_1) is minimal.

Note that (A5) actually strengthens the customary SP assumption which only requires that (A, B_1) and (C_1, A) have no uncontrollable and unobservable mode on the imaginary axis, respectively. This is meant to ensure minimality of the closed-loop system in order to use the simplest version of the Bounded Real Lemma. For extensions to the general case, see [7].

In approaches based on the Bounded Real Lemma, the first step consists of introducing a state-space realization of the closed-loop system $\mathcal{F}(G, K)$. To this purpose, consider a minimal realization of the controller K :

$$K(s) = D_K + C_K(sI - A_K)^{-1}B_K; \quad A_K \in \mathbb{R}^{k \times k}. \quad (2.4)$$

Throughout the paper we restrict our attention to controllers of order k less than or equal to the plant order n . Note that reduced-order controllers ($k < n$) are therefore included in our approach. Given (2.4), a realization of $\mathcal{F}(G, K)$ easily follows as:

$$\mathcal{F}(G, K)(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl} \quad (2.5)$$

where

$$\begin{aligned} A_{cl} &= \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{pmatrix}; & B_{cl} &= \begin{pmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{pmatrix}; \\ C_{cl} &= (C_1 + D_{12} D_K C_2 \quad D_{12} C_K); & D_{cl} &= D_{12} D_K D_{21}. \end{aligned} \quad (2.6)$$

Note that **(A5)** and the minimality of (A_K, B_K, C_K) ensure the minimality of this realization [15].

By invoking the Bounded Real Lemma, the H_∞ constraints on $\mathcal{F}(G, K)$ can be replaced by conditions on a Riccati equation involving the closed-loop parameters $A_{cl}, B_{cl}, C_{cl}, D_{cl}$. This lemma provides the fundamental link between $\mathcal{P}_\infty(\gamma)$ and Riccati equations. It is now briefly recalled in its simpler form, that is, for strictly proper and minimal realizations. The proof appears in [1].

Lemma 2.1 (Bounded Real Lemma) *Provided that (A, B, C) is minimal with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, there is equivalence between :*

- A is stable and $\|C(sI - A)^{-1}B\|_\infty \leq \gamma$.
- There exists a symmetric **positive definite** solution X to the algebraic Riccati equation

$$A^T X + X A + X B B^T X + \gamma^{-2} C^T C = 0. \quad (2.7)$$

Appropriate extensions of this basic version are needed to handle the non strictly proper case as well as problems where **(A5)** is relaxed so that the closed-loop realization (2.5) may be non minimal. For simplicity, results will be stated and complete proofs will be given only for cases where this simple version suffices.

3 Outline of the Proposed Solution

From Lemma 2.1, a strictly proper controller $K(s) = C_K(sI - A_K)^{-1}B_K$ solves $\mathcal{P}_\infty(\gamma)$ if and only if there exists $X_{cl} = X_{cl}^T > 0$ solving the Riccati equation:

$$A_{cl}^T X_{cl} + X_{cl} A_{cl} + X_{cl} B_{cl} B_{cl}^T X_{cl} + \gamma^{-2} C_{cl}^T C_{cl} = 0 \quad (3.1)$$

where A_{cl}, B_{cl}, C_{cl} are given by (2.6) with $D_K = 0$. Note that X_{cl} is not unique in general.

This section informally shows how (3.1) leads to a pair of GRE's which closely resemble the usual H_∞ GRE's. In their general form, these GRE's have an additional term which involves the parameters B_K and C_K of the controller. If K is chosen to be the central controller however, this additional term vanishes and we recover the usual GRE's associated with $\mathcal{P}_\infty(\gamma)$ [5]:

$$A^T X_\infty + X_\infty A + X_\infty (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty + C_1^T C_1 = 0; \quad (3.2)$$

$$A Y_\infty + Y_\infty A^T + Y_\infty (\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y_\infty + B_1 B_1^T = 0. \quad (3.3)$$

In the sequel K is assumed of order $k \leq n$ where n is the order of the plant.

Suppose K is a particular solution of $\mathcal{P}_\infty(\gamma)$ so that (3.1) has some solution $X_{cl} > 0$. Partition X_{cl} conformably to A_{cl} as $X_{cl} = \begin{pmatrix} S & \bar{N} \\ \bar{N}^T & \psi \end{pmatrix}$. With (A1)-(A5) standing and $D_K = 0$, (3.1) is equivalent to:

$$A^T S + SA + SB_1 B_1^T S + \gamma^{-2} C_1^T C_1 - C_2^T C_2 + (C_2^T + \bar{N} B_K)(C_2^T + \bar{N} B_K)^T = 0; \quad (3.4)$$

$$(A + B_1 B_1^T S)^T \bar{N} + \bar{N} A_K + SB_2 C_K + (C_2^T + \bar{N} B_K) B_K^T \psi = 0; \quad (3.5)$$

$$A_K^T \psi + \psi A_K + \psi B_K B_K^T \psi + \gamma^{-2} C_K^T C_K + C_K^T B_2^T \bar{N} + \bar{N}^T B_2 C_K + \bar{N}^T B_1 B_1^T \bar{N} = 0. \quad (3.6)$$

Defining $Y := S^{-1}$ and $N := Y \bar{N}$, the first equation can be rewritten:

$$AY + YA^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2)Y + B_1 B_1^T + (YC_2^T + NB_K)(YC_2^T + NB_K)^T = 0 \quad (3.7)$$

which is exactly the usual “observer” GRE (3.3) except for the additional term $(YC_2^T + NB_K)(YC_2^T + NB_K)^T$.

To obtain the counterpart of the “controller” GRE (3.2), observe that (3.2) and (3.3) are dual when replacing the plant $G(s)$ by $G^T(s)$. This suggests applying the same argument to the transposed closed-loop system $\mathcal{F}(G^T, K^T) = B_{cl}^T(sI - A_{cl}^T)^{-1} C_{cl}^T$. In turn, this amounts to replacing (3.1) by its dual

$$A_{cl} Y_{cl} + Y_{cl} A_{cl}^T + Y_{cl} C_{cl}^T C_{cl} Y_{cl} + \gamma^{-2} B_{cl} B_{cl}^T = 0 \quad (3.8)$$

while now requiring existence of a solution $Y_{cl} = Y_{cl}^T > 0$ to this equation. Not surprisingly because of duality, such a solution is readily available as $Y_{cl} = \gamma^{-2} X_{cl}^{-1}$ where X_{cl} is the positive definite solution of (3.1) introduced above. Partitioning $\gamma^{-2} X_{cl}^{-1}$ as $\begin{pmatrix} R & \bar{M} \\ \bar{M}^T & \theta \end{pmatrix}$, writing that $\gamma^{-2} X_{cl}^{-1}$ solves (3.8) and looking at the (1,1) block then yields:

$$AR + RA^T + RC_1^T C_1 R + \gamma^{-2} B_1 B_1^T - B_2 B_2^T + (B_2 + \bar{M} C_K^T)(B_2 + \bar{M} C_K^T)^T = 0 \quad (3.9)$$

or equivalently

$$A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X + C_1^T C_1 + (XB_2 + \bar{M} C_K^T)(XB_2 + \bar{M} C_K^T)^T = 0 \quad (3.10)$$

upon defining $X := R^{-1}$ and $M := X \bar{M}$. Again this equation differs from (3.2) only by the additional rightmost squared term.

Summing up, if $K = C_K(sI - A_K)^{-1} B_K$ solves $\mathcal{P}_\infty(\gamma)$ then the GRE's (3.7) and (3.10) are solvable for Y and X . Moreover, it is easily verified that $X_{cl} > 0$ together with $Y_{cl} X_{cl} = \gamma^{-2} I$ implies that

$$MN^T = \gamma^{-2} XY - I \quad (3.11)$$

and that $R > 0$, $S > 0$, and $\lambda_{\min}(RS) \geq \gamma^{-2}$. Note that these three conditions can be rewritten in the more familiar form:

$$X > 0; \quad Y > 0; \quad \rho(XY) \leq \gamma^2. \quad (3.12)$$

Finally, (3.7) and (3.10) reduce to the usual GRE's when

$$XB_2 + MC_K^T = 0; \quad YC_2^T + NB_K = 0. \quad (3.13)$$

Taking $X = X_\infty$ and $Y = Y_\infty$, (3.5) and (3.13) can then be solved for A_K, B_K, C_K and this yields the central controller formulas (see Section 5). Hence, the usual GRE's (3.2)-(3.3) are a special case of (3.10) and (3.7) associated with the central controller.

Conversely, we may wonder to which extent the GRE's (3.7) and (3.10) and conditions like (3.11)-(3.12) characterize suboptimality of a controller. In other words, given B_K, C_K and X, Y, M, N satisfying these equations and conditions, can we always reconstruct some suboptimal controller? In light of the previous argument, this amounts to reconstructing the controller state matrix A_K and $X_{cl} > 0$ solving (3.1). Now, observe that X_{cl} is entirely determined by X, Y, M, N using the previous block partitions. As of A_K , it must be chosen to satisfy (3.5). The suboptimality of the resulting controller is then a matter of verification.

Despite its lack of rigor, this section has exposed most of the connection between $\mathcal{P}_\infty(\gamma)$ and the H_∞ Riccati equations (3.2)-(3.3). Insight has also been gained into the foundation of the suboptimal controller representation developed in this paper. These principles are now mathematically formalized in the next two sections.

4 A New Representation of H_∞ Controllers

A new representation is proposed for H_∞ suboptimal controllers of order no larger than the plant order. This representation associates with each controller a pair of game Riccati equations (GRE). These equations are very similar to the usual GRE's except for an additional term which involves the controller parameters and thus explicitly reflects the controller specificity. Solutions of these modified GRE's are also constrained by the familiar positivity and spectral radius conditions. Finally, there is some extra coupling condition in the reduced-order case. As a direct consequence of the Bounded Real Lemma, this representation is conceptually straightforward. Yet, it summarizes in a very condensed manner what makes a controller suboptimal and should therefore be a powerful tool to probe and exploit the suboptimal controller diversity.

As mentioned earlier, the argument is specialized to the Standard Problem and (A, B_1, C_1) is assumed to be a minimal realization of $G_{11}(s)$. The main result is first enunciated for strictly proper controllers and then extended to the non strictly proper case. All proofs are deferred to Appendix A.

Theorem 4.1 (Representation of strictly proper suboptimal controllers)

With assumptions (A1)-(A5),

(i) if the controller K of order $k \leq n$ solves $\mathcal{P}_\infty(\gamma)$ then given the minimal realization:

$$K(s) = C_K(sI - A_K)^{-1}B_K; \quad A_K \in \mathbf{R}^{k \times k}, \quad (4.1)$$

there exist full rank matrices M, N in $\mathbf{R}^{n \times k}$ and symmetric $n \times n$ matrices X, Y such that:

(C1) X, Y, M , and N satisfy

$$A^T X + X A + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 + (X B_2 + M C_K^T)(X B_2 + M C_K^T)^T = 0; \quad (4.2)$$

$$A Y + Y A^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T + (Y C_2^T + N B_K)(Y C_2^T + N B_K)^T = 0; \quad (4.3)$$

$$M N^T = \gamma^{-2} X Y - I. \quad (4.4)$$

(C2) $X > 0$, $Y > 0$, and $\rho(XY) \leq \gamma^2$.

(ii) Conversely, consider any B_K, C_K, X, Y, M, N which jointly satisfy (C1)-(C2). Then if k denotes the rank of $\gamma^{-2}XY - I$, B_K, C_K, M, N can always be redefined so that M, N are full

rank $n \times k$ matrices. With this new notation, let $A_K \in \mathbb{R}^{k \times k}$ be the unique solution to

$$N A_K M^T = (\gamma^{-2} Y X - I) \{A + \gamma^{-2} B_1 B_1^T X + B_2 C_K M^T\} - N B_K C_2 + \gamma^{-2} Y M C_K^T (B_2^T X + C_K M^T) \quad (4.5)$$

or alternatively,

$$N A_K M^T = \{A + \gamma^{-2} Y C_1^T C_1 + N B_K C_2\} (\gamma^{-2} Y X - I) - B_2 C_K M^T + \gamma^{-2} (Y C_2^T + N B_K) B_K^T N^T X \quad (4.6)$$

$$= -(A + \gamma^{-2} Y A^T X) - (\gamma^{-2} B_1 B_1^T X + B_2 C_K M^T) - (\gamma^{-2} Y C_1^T C_1 + N B_K C_2). \quad (4.7)$$

Then $K(s) := C_K(sI - A_K)^{-1} B_K$ is a k th order solution to $\mathcal{P}_\infty(\gamma)$. ■

This central result warrants a few comments which are now gathered in a series of remarks. Their aim is to give further insight into the nature, structure and interpretation of the new representation.

Remark 4.2 (Interpretation of (C1)-(C2)): The application of Lemma 2.1 to the closed-loop system yields the Riccati equation (3.1). The matrices X, Y, M, N are then directly extracted from solutions X_{cl} of this equation. Hence, existence of X, Y, M, N satisfying (C1) is equivalent to solvability of (3.1) and therefore expresses that $\|\mathcal{F}(G, K)\|_\infty \leq \gamma$. Meanwhile, (C2) is equivalent to $X_{cl} > 0$ and therefore enforces internal stability.

Remark 4.3 (Necessity part): The equations in (C1) must have solutions satisfying (C2) if the controller K of order $k \leq n$ and minimal realization (4.1) is suboptimal. Note that these equations are specialized to K through B_K and C_K . Yet, X, Y, M, N are not unique in general since they come from solutions of the Bounded Real Lemma equation (3.1). Hence, one suboptimal controller yields several quadruples (X, Y, M, N) or equivalently, several GRE's (4.2)-(4.3) corresponding to all possible (M, N) pairs. In that sense, this representation is not one-to-one.

Remark 4.4 (Sufficiency part): Conversely, each solution of (C1)-(C2) characterizes one suboptimal controller whose state matrix A_K is given by (4.5). Even though six unknowns B_K, C_K, X, Y, M, N must be determined instead of just A_K, B_K, C_K , suboptimality is now captured in a much more structured and tractable way. Indeed, full-order solutions of (C1)-(C2) are easily generated as follows:

- consider any pair of matrices L_X and L_Y of the same dimensions as $X B_2 + M C_K^T$ and $Y C_2^T + N B_K$, and such that (4.2)-(4.3) are solvable when $X B_2 + M C_K^T$ and $Y C_2^T + N B_K$ are replaced by L_X and L_Y , respectively. Tune L_X and L_Y so that $X \geq 0, Y \geq 0$, and $\rho(XY) < \gamma^2$.
- Pick M, N as arbitrary nonsingular factors of $\gamma^{-2}XY - I$.
- Deduce B_K, C_K by solving $X B_2 + M C_K^T = L_X$ and $Y C_2^T + N B_K = L_Y$, and A_K by solving (4.5).

Remark 4.5 (Controller order): Provided that (A_K, B_K, C_K) is minimal, the controller order is equal to the rank of $\gamma^{-2}XY - I$ or equivalently, to the column dimension k of the full rank matrices M and N . Reduced-order suboptimal controllers ($k < n$) are therefore characterized by the saturation of the constraint $\rho(XY) \leq \gamma^2$, that is, by $\rho(XY) = \gamma^2$. By contrast, solutions for which $\rho(XY) < \gamma^2$ yield full-order ($k = n$) controllers among which the usual central controller.

Remark 4.6 (Coupling equation): Solving (4.2)-(4.4) may seem untractable at first because of the coupling constraint (4.4) which binds the two GRE's (4.2)-(4.3) via M and N . Yet, this constraint turns out to play no role for full-order controllers ($k = n$). Indeed, M and N are then square invertible and can be "absorbed" in B_K and C_K . That is, given B_K, C_K, X, Y, M, N solving (4.2)-(4.3) we can always redefine M, N, B_K, C_K so that (4.4) is satisfied. In this case moreover, the resulting controller is independent of the particular choice of M, N provided (4.4) is satisfied. Indeed, changing M simply amounts to a coordinate transformation on the controller state. By contrast, (4.4) becomes nontrivial for reduced-order controllers ($k < n$) which confirms the incremental difficulty of reduced-order design.

This section is concluded with an extension of the representation of Theorem 4.1 to proper controllers. The proof is omitted for brevity (see [7] for details).

Theorem 4.7 (Representation of proper suboptimal controllers)

Theorem 4.1 remains valid for the larger class of proper controllers of order $k \leq n$:

$$K(s) = D_K + C_K(sI - A_K)^{-1}B_K \quad (4.8)$$

with the following modifications

- D_K is constrained to $\sigma_{\max}(D_K) \leq \gamma$.
- (4.2) - (4.3) must be replaced with:

$$\begin{aligned} A^T X + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^T C_1 + \\ (XB_2 + MC_K^T + C_2^T D_K^T)(I - \gamma^{-2}D_K D_K^T)^{-1}(XB_2 + MC_K^T + C_2^T D_K^T)^T = 0; \end{aligned} \quad (4.9)$$

$$\begin{aligned} AY + Y A^T + Y(\gamma^{-2}C_1^T C_1 - C_2^T C_2)Y + B_1 B_1^T + \\ (YC_2^T + NB_K + B_2 D_K)(I - \gamma^{-2}D_K^T D_K)^{-1}(YC_2^T + NB_K + B_2 D_K)^T = 0; \end{aligned} \quad (4.10)$$

- (4.5) becomes:

$$\begin{aligned} NA_K M^T = - \{ A + B_2 D_K C_2 + \gamma^{-2}(Y A^T X + B_1 B_1^T X + Y C_1^T C_1) + B_2 C_K M^T + NB_K C_2 \} - \\ (YC_2^T + B_2 D_K + NB_K)(I - \gamma^{-2}D_K^T D_K)^{-1}D_K^T (B_2^T X + D_K C_2 + C_K M^T). \end{aligned} \quad (4.11)$$

- K in (ii) is now defined as $K(s) := D_K + C_K(sI - A_K)^{-1}B_K$.

Remark 4.8 The GRE's (4.9)-(4.10) retain the same structure as when $D_K = 0$. Note that if $\sigma_{\max}(D_K) = \gamma$, the matrices $\gamma^{-2}I - D_K^T D_K$ and $\gamma^{-2}I - D_K D_K^T$ are no longer invertible. The equation (4.9) should then read: there exists X and Γ satisfying the system of equations:

$$\begin{aligned} A^T X + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^T C_1 + (XB_2 + MC_K^T + C_2^T D_K) \Gamma = 0; \\ (I - \gamma^{-2}D_K D_K^T) \Gamma - (XB_2 + MC_K^T + C_2^T D_K)^T = 0, \end{aligned} \quad (4.12)$$

and similarly for (4.10).

5 Recovering DGKF's Solvability Conditions

This section relates the results of Theorems 4.1 and 4.7 to the classical solvability conditions of [5]. Recall that under assumptions (A1)-(A5), [5] establishes equivalence of the following statements:

- (1) there exists an internally stabilizing controller $K(s)$ such that $\|\mathcal{F}(G, K)\|_\infty < \gamma$;
- (2) the GRE's (3.2)-(3.3) have real symmetric stabilizing solutions X_∞ and Y_∞ satisfying $X_\infty \geq 0$, $Y_\infty \geq 0$, and $\rho(X_\infty Y_\infty) < \gamma^2$.

Hence, it suffices for solvability of $\mathcal{P}_\infty(\gamma)$ to secure solvability of one particular problem (C1)-(C2), or equivalently to secure existence of one particular controller: the “central” controller obtained from (4.2)-(4.3) when imposing $XB_2 + MC_K^T = 0$, $YC_2^T + NB_K = 0$, and X, Y stabilizing. This claim is surprising at first but happens to have a simple explanation. Indeed, the stabilizing solutions X_∞ and Y_∞ of (3.2)-(3.3) turn out to be extremal among all possible solutions X, Y compatible with (C1)-(C2). In loose terms, this stems from the fact that X_∞ and Y_∞ are stabilizing and zero the rightmost nonnegative terms in (4.9)-(4.10). In turn this extremality property readily accounts for the special role played by these two solutions when characterizing solvability of $\mathcal{P}_\infty(\gamma)$.

The following lemma is instrumental to establishing the minimality property of X_∞ and Y_∞ .

Lemma 5.1 (Monotonicity and minimality properties for H_∞ GRE's)

With (A, B) stabilizable and $(C, -A)$ detectable, assume that the GRE

$$A^T X_0 + X_0 A + X_0 (P_0 - BB^T) X_0 + C^T C + Q_0 = 0 \quad (5.1)$$

has a symmetric solution $X_0 > 0$ for some $P_0 \geq 0$ and $Q_0 \geq 0$. Then for any P, Q satisfying $0 \leq P \leq P_0$ and $0 \leq Q \leq Q_0$, the GRE

$$A^T X + X A + X (P - BB^T) X + C^T C + Q = 0 \quad (5.2)$$

has a **strong** solution $X(P, Q)$, that is, a symmetric solution such that the closed-loop matrix $A + (P - BB^T)X$ has all its eigenvalues in the closed left-half plane. Moreover,

- $0 < X(P, Q) \leq X_0$ and $X(P, Q)$ is therefore minimal for $P = Q = 0$.
- $X(P, Q)$ is stabilizing if and only if

$$A^T X + X A + X \left\{ (1 + \epsilon)P - BB^T \right\} X + C^T C + Q = 0 \quad (5.3)$$

retains a positive definite solution for some $\epsilon > 0$.

Proof : See Appendix B. ■

The equivalence of (1) and (2) is now derived as a consequence of Theorems 4.1 and 4.7. Observe that since (A, B_1, C_1) is assumed minimal, X_∞ and Y_∞ are nonsingular so that nonnegativity means in fact positive definiteness in (2).

(1) \Rightarrow (2): Suppose (1) holds. Then from (i) of Theorem 4.7, there exist matrices $X > 0$ and $Y > 0$ which solve (4.9)-(4.10). Since (A, B_2) and $(C_1, -A)$ are assumed stabilizable and detectable,

respectively, the monotonicity result of Lemma 5.1 can be applied with $P = P_0 := \gamma^{-2} B_1 B_1^T$, $Q := 0$, and

$$Q_0 := (X B_2 + M C_K^T + C_2^T D_K^T) (I - \gamma^{-2} D_K D_K^T)^{-1} (X B_2 + M C_K^T + C_2^T D_K^T)^T.$$

It follows that the GRE

$$A^T X + X A + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 = 0 \quad (5.4)$$

has a positive definite strong solution $X(P, 0)$. Now observe that γ could be replaced by $\gamma - \epsilon$ for small enough $\epsilon > 0$ in virtue of the strict inequality in (1). Since this amounts to multiplying P by a number greater than 1, Lemma 5.1 guarantees that $X(P, 0)$ is in fact stabilizing. Hence, (5.4) has a stabilizing solution $X_\infty := X(P, 0)$ such that $0 < X_\infty \leq X$.

Similarly, solvability of (4.10) for some $Y > 0$ ensures that (3.3) has a stabilizing solution Y_∞ satisfying $0 < Y_\infty \leq Y$. Finally,

$$\rho(X_\infty Y_\infty) = \rho(X_\infty^{1/2} Y_\infty X_\infty^{1/2}) \leq \rho(X_\infty^{1/2} Y X_\infty^{1/2}) = \rho(Y^{1/2} X_\infty Y^{1/2}) \leq \rho(Y^{1/2} X Y^{1/2}) = \rho(X Y)$$

which together with (C2) gives $\rho(X_\infty Y_\infty) \leq \gamma^2$. This inequality can actually be strengthened by again observing that $\|\mathcal{F}(G, K)\|_\infty \leq \gamma - \epsilon$ for some $\epsilon > 0$. Denoting by X_∞^ϵ and Y_∞^ϵ the counterparts of X_∞ and Y_∞ when $\gamma - \epsilon$ replaces γ , Lemma 5.1 ensures that $X_\infty \leq X_\infty^\epsilon$ and $Y_\infty \leq Y_\infty^\epsilon$ while $\rho(X_\infty^\epsilon Y_\infty^\epsilon) \leq (\gamma - \epsilon)^2$ from the previous argument. Consequently, $\rho(X_\infty Y_\infty) < \gamma^2$.

(2) \Rightarrow (1): A particular solution of (C1)-(C2) in Theorem 4.1 is easily reconstructed from X_∞ and Y_∞ . Indeed, it suffices to chose B_K, C_K, M, N so that

$$X_\infty B_2 + M C_K^T = 0; \quad Y_\infty C_2^T + N B_K = 0; \quad M N^T = \gamma^{-2} X_\infty Y_\infty - I. \quad (5.5)$$

Recalling that the choice of (M, N) compatible with (4.4) is immaterial, select

$$M := I; \quad N := \gamma^{-2} Y_\infty X_\infty - I. \quad (5.6)$$

Here N is invertible since $\rho(X_\infty Y_\infty) < \gamma^2$ and this choice together with (4.5) and (5.5) entirely determines A_K, B_K, C_K as:

$$\begin{aligned} A_{K_c} &= A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty - (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty C_2^T C_2; \\ B_{K_c} &= -N^{-1} Y_\infty C_2^T = (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty C_2^T; \quad C_{K_c} = -B_2^T X_\infty M^{-T} = -B_2^T X_\infty. \end{aligned} \quad (5.7)$$

With these parameters, $X := X_\infty$, and $Y := Y_\infty$, (C1)-(C2) is satisfied and $K_c := C_{K_c}(sI - A_{K_c})^{-1} B_{K_c}$ is therefore a solution of $\mathcal{P}_\infty(\gamma)$ in virtue of Theorem 4.1, part (ii). Note that (5.7) coincide with the central controller formulas of [5]. Hence, $(B_{K_c}, C_{K_c}, X_\infty, Y_\infty, I, \gamma^{-2} Y_\infty X_\infty - I)$ is a particular solution of (C1)-(C2) which characterizes the central controller.

The proof is complete upon strengthening $\|\mathcal{F}(G, K)\|_\infty \leq \gamma$ to a strict inequality. This is done as earlier by observing that stabilizing solutions of Riccati equations are retained in the vicinity of the equation parameters and depend continuously on these parameters [4]. Hence the same argument carries through with γ replaced by $\gamma - \epsilon$ for $\epsilon > 0$ small enough. ■

This section is concluded with a characterization of solvability at the optimum when γ_{opt} .

Theorem 5.2 *In the Standard Problem framework, and provided that $\gamma_{opt} > 0$, the following statements are equivalent:*

(1) there exists an internally stabilizing controller $K(s)$ such that $\|\mathcal{F}(G, K)\|_\infty = \gamma_{opt}$;

(2) the GRE's (3.2)-(3.3) with $\gamma = \gamma_{opt}$ have strong solutions X_∞ and Y_∞ satisfying $X_\infty \geq 0$, $Y_\infty \geq 0$, and $\rho(X_\infty Y_\infty) \leq \gamma_{opt}^2$.

Proof: The necessity of (2) is established by the same argument as above except for two differences: strong can no longer be strengthened to stabilizing and the inequality $\rho(X_\infty Y_\infty) \leq \gamma_{opt}^2$ may be an equality.

The converse requires more care since (5.7) is not always meaningful at $\gamma = \gamma_{opt}$. In most cases indeed, $N = \gamma_{opt}^{-2}I - Y_\infty^{-1}X_\infty^{-1}$ is singular. Nevertheless, this difficulty can be circumvented by reconstructing a non strictly proper “generalized” central controller. Such controllers are derived from (4.9)-(4.10) instead of (4.2)-(4.3) and obtained by imposing

$$XB_2 + MC_K^T + C_2^T D_K^T = 0; \quad YC_2^T + NB_K + B_2 D_K = 0$$

while taking $X := X_\infty$ and $Y := Y_\infty$ (see [8] for details). Theorem 4.7 then ensures that the resulting controller solves $P_\infty(\gamma_{opt})$. ■

6 Advantages of the New Representation

This section compares the representation in Section 4 with the Q -parametrization of all H_∞ -suboptimal controllers of [5]. Interestingly, these two representations turn out to be quite different in spirit and applicability. For a worthwhile comparison of the two approaches, the form and principle of the Q -parametrization are first recalled. This representation is built around the central controller K_c (see Section 5) and uses a LFT to parametrize all solutions K of $\mathcal{P}_\infty(\gamma)$ as $K = \mathcal{F}(J_c, Q)$ where \mathcal{F} is defined in (2.2), J_c is an embedding of K_c defined as

$$J_c := \left(\begin{array}{c|cc} \frac{A_{K_c}}{-B_2^T X_\infty} & (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty C_2^T & (I - \gamma^{-2} Y_\infty X_\infty)^{-1} B_2 \\ \hline & 0 & I \\ & I & 0 \end{array} \right) \quad (A_{K_c} \text{ as in (5.7)}), \quad (6.1)$$

and Q is a free parameter in the ball

$$\mathcal{BR}_\infty^{m_2 \times p_2}(\gamma) := \{Q \text{ has stable poles and } \|Q\|_\infty < \gamma\}. \quad (6.2)$$

Note that the center $Q \equiv 0$ of the ball (6.2) yields $\mathcal{F}(J_c, 0) = K_c$ whence the term “central” controller. Also, J_c represents the feedback connections between Q and K_c , that is, how the dynamics of Q and K_c are coupled.

The Q -parametrization proceeds from the following abstract property of LFT's.

Lemma 6.1 Consider any strictly proper transfer function $F_0(s)$ in $\mathcal{BR}_\infty(\gamma)$ and introduce the following embedding of F_0 :

$$\Gamma_{F_0} := \left(\begin{array}{c|cc} A_0 & B_0 & -\gamma^{-2} X_0^{-1} C_0^T \\ \hline C_0 & 0 & I \\ -B_0^T X_0 & I & 0 \end{array} \right) \quad (6.3)$$

where (A_0, B_0, C_0) is a minimal realization of F_0 and $X_0 = X_0^T > 0$ is the (unique) stabilizing solution of the Bounded Real Lemma equation:

$$A_0^T X_0 + X_0 A_0 + X_0 B_0 B_0^T X_0 + \gamma^{-2} C_0^T C_0 = 0. \quad (6.4)$$

This particular embedding has the following properties:

- the closed-loop system $F(Q) := \mathcal{F}(\Gamma_{F_0}, Q)$ is internally stable and satisfies $\|F(Q)\|_\infty < \gamma$ if and only if $Q \in \mathcal{BR}_\infty(\gamma)$;
- Γ_{F_0} is all-pass for $\gamma = 1$. That is, $\Gamma_{F_0}^T(-s)\Gamma_{F_0}(s) = I$ for all $s \in \mathbb{C}$.

Proof: See Appendix C. ■

Given any controller K_0 solving $\mathcal{P}_\infty(\gamma)$, this lemma can be applied to $F_0 := \mathcal{F}(G, K_0)$. This yields a LFT parametrization of all stable transfer functions F from w to z such that $\|F\|_\infty < \gamma$. This parametrization is represented in Figure 6.2a. To deduce the Q -parametrization of all suboptimal controllers, we need to rearrange the interconnections between G, K_0 , and Q so as to lump K_0 and Q into a single controller acting on G as in Figure 6.2b. Of course, such restructuring is not possible in general. Nevertheless, there are special choices of K_0 which allow to decouple the dynamics of Q and G so that K_0 and Q can be lumped together. Not surprisingly, the central controller K_c is among such choices. Evidence for these claims is gathered in the following state-space argument.

To begin with, introduce a minimal realization $K_0(s) = C_K(sI - A_K)^{-1}B_K$ of K_0 and use (2.6) as realization of $F_0 := \mathcal{F}(G, K_0)$ (with $D_K = 0$). A possible choice for X_0 of Lemma 6.1 is then $X_{cl} = \begin{pmatrix} S & N \\ \bar{N}^T & \psi \end{pmatrix}$ of Section 3. Recalling that $\gamma^{-2}X_{cl}^{-1} = \begin{pmatrix} R & M \\ \bar{M}^T & \theta \end{pmatrix}$, and given a (minimal) realization $Q(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B}$ of Q , a realization of $F(Q) := \mathcal{F}(\Gamma_{F_0}, Q)$ is obtained as:

$$F(Q) = \left(\begin{array}{ccc|c} A & B_2 C_K & -(\bar{M} C_K^T D_{12}^T + R C_1^T) \bar{C} & B_1 \\ B_K C_2 & A_K & -(\theta C_K^T D_{12}^T + \bar{M}^T C_1^T) \bar{C} & B_K D_{21} \\ \hline -\bar{B}(D_{21}^T B_K^T \bar{N}^T + B_1^T S) & -\bar{B}(D_{21}^T B_K^T \psi + B_1^T \bar{N}^T) & \bar{A} & \bar{B} \\ C_1 & D_{12} C_K & \bar{C} & 0 \end{array} \right). \quad (6.5)$$

In comparison, for a generic controller $K = U(sI - W)^{-1}V$ the realization of $\mathcal{F}(G, K)$ has the form:

$$\left(\begin{array}{cc|c} A & B_2 U & B_1 \\ V C_2 & W & V D_{21} \\ \hline C_1 & D_{12} U & 0 \end{array} \right). \quad (6.6)$$

Consequently, (6.5) can be interpreted as a single controller K acting on G if and only if

- Q is of the form $Q(s) = D_{12} Q_r(s) D_{21}$ so that \bar{B} and \bar{C} are of the form $V D_{21}$ and $D_{12} U$, respectively.
- $-(\bar{M} C_K^T D_{12}^T + R C_1^T) \bar{C} = B_2 \bar{C}$ and $-\bar{B}(D_{21}^T B_K^T \bar{N}^T + B_1^T S) = \bar{B} C_2$.

Combining (a) and (b), decoupling will therefore occur as soon as $X B_2 + M C_K^T = 0$, $Y C_2^T + N B_K = 0$, and $Q = D_{12} Q_r D_{21}$. Hence, only those controllers obtained by zeroing the quadratic tail terms in (4.2)-(4.3) achieve full decoupling of Q and G , among which the central controller K_c .

Set $K_0 := K_c$ and write $\bar{B} = \bar{B}_r D_{21}$ and $\bar{C} = D_{12} \bar{C}_r$ so that $Q(s)$ is of the form $D_{12} Q_r(s) D_{21}$. After simplification using (A3) and (5.7), (6.5) confirms that $F(Q) := \mathcal{F}(\Gamma_{F_0}, Q)$ can be rewritten as $\mathcal{F}(G, K)$ where

$$K = \left(\begin{array}{cc|c} A_{K_c} & -\theta C_{K_c}^T \bar{C}_r & B_{K_c} \\ -\bar{B}_r B_{K_c}^T \psi & \bar{A} & \bar{B}_r \\ \hline C_{K_c} & \bar{C}_r & 0 \end{array} \right) = \left(\begin{array}{cc|c} A_{K_c} & (I - \gamma^{-2} Y_\infty X_\infty)^{-1} B_2 \bar{C}_r & B_{K_c} \\ -\bar{B}_r C_2 & \bar{A} & \bar{B}_r \\ \hline C_{K_c} & \bar{C}_r & 0 \end{array} \right) = \mathcal{F}(J_c, Q_r). \quad (6.7)$$

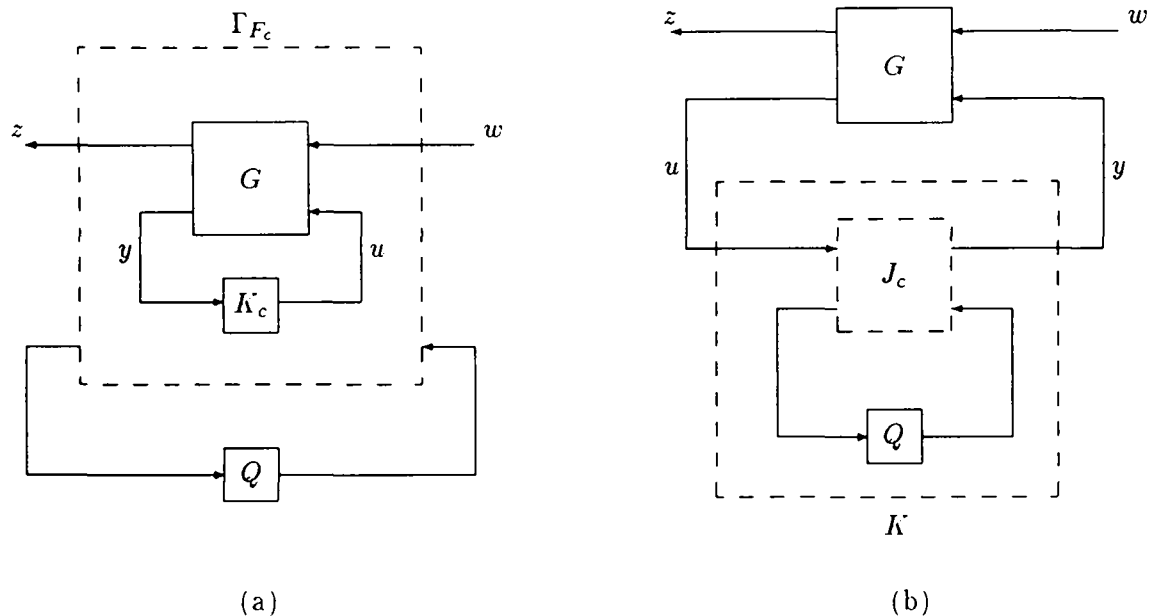


Figure 6.2

In virtue of Lemma 6.1, all solutions of $\mathcal{P}_\infty(\gamma)$ are therefore of the form $K = \mathcal{F}(J_c, Q_r)$ where $Q_r \in \mathcal{BR}_\infty^{m_2 \times p_2}(\gamma)$. Finally, note that for Γ_{F_0} is all-pass for $K_0 = K_c$ and $\gamma = 1$, a fact at the origin of the all-pass embedding approach of [10].

This original perspective gives much insight into the strengths and limitations of the Q -parametrization. On one hand it offers a mathematically concise and exhaustive description of suboptimal controllers. On the other hand, additional objectives and constraints on the closed-loop response are difficult to monitor through the choice of Q . In addition, there is no direct access to the controller true dynamics and reduced-order controllers are not easily retrievable. Indeed, a controller K of order $k < n$ is represented as $\mathcal{F}(J_c, Q)$ where J_c is of order n and Q is of order $r \geq 0$. That is, by a system of generic order $n + r$ where $n + r - k$ pole/zero cancellations must occur. This is of course impractical from a numerical standpoint.

By contrast, the representation (C1)-(C2) is not exhaustive since controllers of order larger than the plant order are not fully described. However, it offers direct access to the controller dynamics via A_K, B_K, C_K and does not involve any pole/zero cancellations to represent reduced-order controllers. Moreover, its algebraic formulation and state-space framework are propitious to analytical and numerical treatment of more complex problems where other criterion constrain the choice of controller. Examples of problems which nicely blend into this framework are minimum-entropy design, reduced-order design, mixed H_2/H_∞ , etc. For these reasons, (C1)-(C2) has a promising potential for investigating and exploiting the diversity of H_∞ -suboptimal controllers.

7 An Example

The usage and benefits of this representation are illustrated on the following elementary example.

Consider the plant $G = \left(\begin{array}{c|cc} \delta & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \right)$ where δ is a small real number. This plant is associated with the equations:

$$\dot{x} = \delta x + u; \quad z = u; \quad y = x + w.$$

Here the optimal gain is given by $\gamma_{opt} = 2 \max(\delta, 0)$. For $\delta = 0$, the H_∞ problem is “singular” since zero is then a nonminimal mode of (A, B_1, C_1) . For $\delta < 0$ the central controller K_c is identically zero and for $\delta > 0$

$$K_c(s) = -\frac{4\delta^2\gamma^2}{(\gamma^2 - 4\delta^2)s + \delta(3\gamma^2 - 4\delta^2)}.$$

Note that closed-loop modes are of order δ so that internal stability is only marginal with the central controller.

Consider now the family of equations (4.2)-(4.3). In terms of $\alpha = xb_2 + mc_K$ and $\beta = yc_2 + nb_K$ they read

$$2\delta x - x^2 + \alpha^2 = 0; \quad 2\delta y - y^2 + \beta^2 = 0.$$

These equations are independent of γ and for all α, β they have nonnegative stabilizing solutions given by $x = \delta + \sqrt{\delta^2 + \alpha^2}$ and $y = \delta + \sqrt{\delta^2 + \beta^2}$. To simplify calculations take $\alpha = \beta$ in the sequel. From Theorem 4.1, a suboptimal controller is obtained as soon as $xy \leq \gamma^2$ which is equivalent to $\alpha^2 \leq (\gamma - \delta)^2 - \delta^2$. This is always feasible since $\gamma \geq \gamma_{opt} = 2 \max(\delta, 0)$.

Take for instance $\gamma > 2\gamma_{opt}$ and chose α so that $x = y = \gamma/2$. Then $m = 1/2$ and $n = -3/4$ are instances compatible with (4.4). In turn, this determines b_K, c_K, a_K via $x + mc_K = y + nb_K = \alpha$. Taking $\alpha < 0$, this yields

$$a_K = \frac{5}{3}\delta - \frac{4}{3}\phi(\gamma, \delta); \quad b_K = \frac{1}{3}\phi(\gamma, \delta); \quad c_K = -\phi(\gamma, \delta) \quad \text{where } \phi(\gamma, \delta) := \gamma + \sqrt{\gamma^2 - 4\gamma\delta}.$$

For $\delta \approx 0$ and γ not too small, $\phi(\gamma, \delta) \approx 2\gamma$ and the resulting controller is approximately $K_\gamma(s) = \frac{-4\gamma^2}{3s+8\gamma}$. Moreover, the closed-loop poles (eigenvalues of A_{cl}) are roughly $-(2 \pm \sqrt{3})\gamma$. Hence this design significantly reinforces internal stability in comparison to the central controller design. Indeed stability margins are now of order γ and will be very small only when approaching the optimal attenuation $\gamma_{opt} \approx 0$.

To calculate reduced-order controllers, observe that such controllers reduce to a static feedback $K(s) = d_K$ due to the plant order $n = 1$. The observer GRE can then be ignored and such K 's solve $\mathcal{P}_\infty(\gamma)$ iff. $\|d_K\| \geq \gamma$ and there exists a nonnegative stabilizing solution X to the GRE (4.9) (with $c_K = 0$). The latter constraint is equivalent to solvability of

$$2\delta x - x^2 + \alpha^2 = 0; \quad x + d_K = \alpha$$

which gives $d_K = \alpha - (\delta + \sqrt{\delta^2 + \alpha^2})$ with α arbitrary. Together with $\|d_K\| \leq \gamma$, this constrains d_K to the interval $[-\gamma, -\delta]$. The closed-loop matrix is $A_{cl} = a + b_2 d_K c_2 = \alpha - \sqrt{\delta^2 + \alpha^2}$ which is always stable. Finally, the choice $K_\gamma = d_K := -\gamma$ yields $\mathcal{F}(G, d_K) = \frac{-\gamma(s-\delta)}{s+\gamma-\delta}$ whose L_∞ norm is no larger than γ since $\gamma \geq \gamma_{opt} \geq 2\delta$. Meanwhile, $A_{cl} = \delta - \gamma \approx -\gamma$. Hence for the specified attenuation γ , this static controller also offers more closed-loop stability than the central controller.

8 Applications and Prospects

The results of Theorem 4.7 have been applied to derive numerically stable formulas for nearly optimal and optimal controllers [8]. Recall that the state-space formulas (5.7) for K_c become ill-conditioned near γ_{opt} whenever $\rho(X_\infty Y_\infty) = \gamma_{opt}^2$ at the optimum. In [8], the following extended notion of central controller is introduced.

Definition 8.1 *We call generalized central controller any solution $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ of $\mathcal{P}_\infty(\gamma)$ determined by (ii) of Theorem 4.7 upon:*

- selecting X and Y as the stabilizing solutions X_∞ and Y_∞ of (3.2)-(3.3);
- imposing

$$X_\infty B_2 + M C_K^T + C_2^T D_K = 0; \quad Y_\infty C_2^T + N B_K + B_2 D_K = 0; \quad (8.1)$$

This definition yields a family a suboptimal controllers parametrized by the feedthrough D_K which is only constrained by $\sigma_{max}(D_K) \leq \gamma$. This fact has been exploited to substantially reduce numerical sensitivity in the computation of A_K, B_K, C_K near γ_{opt} . Specifically, an appropriate choice of D_K allows to neutralize the ill-conditioning attached with the inversion of $I - \gamma^{-2}Y_\infty X_\infty$ near γ_{opt} . The resulting formulas are smooth at γ_{opt} and order reduction involves a finite instead of infinite pole/zero cancellation.

Most current H_∞ design techniques make exclusive use of the central controller. Yet, there are numerous problems where the central controller performs poorly with respect to internal stability, controller complexity, performance, or other criterion of evaluation. It is therefore important to have means of exploring and exploiting the possibilities offered by other suboptimal controllers. In our opinion, the representation introduced in this paper should prove useful with this respect and help incorporating additional objectives and constraints relative to H_2 performance, rising time, internal stability margins, saturations, etc.

Another important prospect is the design of reduced-order controllers. Indeed, the order of K_c typically matches that of the augmented plant which can rapidly become prohibitive in applications such as mixed-sensitivity design or μ synthesis. Of course, *a posteriori* order reduction is always possible but the side-effects of this operation in terms of performance degradation and loss of internal stability are difficult to quantify and monitor. Fixed-order design is therefore of great interest, though a much tougher problem. In [3], this issue is partially addressed in the context of mixed H_2/H_∞ design. The equations characterizing the optimal solutions bear some resemblance with (4.2)-(4.4) despite their higher complexity due to the additional H_2 optimization involved. Coupled with adequate numerical techniques, the simpler characterization of reduced-order controllers given in Theorem 4.1 may prove a powerful tool for H_∞ reduced-order design.

Appendix A

A complete proof of Theorem 4.1 is given under assumptions (A1)-(A5). We begin with a technical lemma.

Lemma A.1 *Consider a nonsingular matrix P and consider the block partitions $P = \begin{pmatrix} R & M \\ M^T & \theta \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} S & N \\ N^T & \psi \end{pmatrix}$. Then $P > 0$ implies that $R > 0$, $S > 0$, and $\lambda_{\min}(RS) \geq 1$, and the converse is true provided that M or N is full column rank.*

Proof: Omitted for brevity. ■

Proof of Theorem 4.1:

Necessity (i): Suppose $K(s)$ of order k solves $\mathcal{P}_\infty(\gamma)$ and consider a minimal realization of K of the form (4.1). The minimality of (A, B_1, C_1) and (A_K, B_K, C_K) ensure that of (A_{cl}, B_{cl}, C_{cl}) Lemma 2.1 then guarantees the existence of a solution $X_{cl} = X_{cl}^T > 0$ to the ARE (3.1). For reasons developed in Section 3, also introduce the solution $Y_{cl} := \gamma^{-2} X_{cl}$ of the dual (3.8) of (3.1). Partition X_{cl} and Y_{cl} as

$$X_{cl} = \begin{pmatrix} S & \bar{N} \\ \bar{N}^T & \psi \end{pmatrix}; \quad Y_{cl} = \begin{pmatrix} R & \bar{M} \\ \bar{M}^T & \theta \end{pmatrix}; \quad \theta, \psi \in \mathbf{R}^{k \times k}, \quad (\text{A.1})$$

and define $X := R^{-1}$, $Y := S^{-1}$, $M := X\bar{M}$, and $N := Y\bar{N}$. From Lemma A.1, $X_{cl} > 0$ ensures that $R > 0$, $S > 0$, and $\lambda_{\min}(RS) \geq \gamma^{-2}$ which is equivalent to (C2). Meanwhile, $Y_{cl}X_{cl} = \gamma^{-2}I$ implies that $\bar{M}\bar{N}^T = \gamma^{-2}I - RS$ which is equivalent to (4.4). Finally, writing the (1,1) block entry of (3.1) and (3.8) yields

$$AR + RA^T + RC_1^T C_1 R + \gamma^{-2} B_1 B_1^T - B_2 B_2^T + (B_2 + \bar{M} C_K^T)(B_2 + \bar{M} C_K^T)^T = 0; \quad (\text{A.2})$$

$$A^T S + SA + SB_1 B_1^T S + \gamma^{-2} C_1^T C_1 - C_2^T C_2 + (C_2^T + \bar{N} B_K)(C_2^T + \bar{N} B_K)^T = 0; \quad (\text{A.3})$$

which are equivalent to (4.2)-(4.3) from the definition of X, Y, M, N .

Sufficiency (ii): Suppose now that B_K, C_K, X, Y, M, N jointly satisfy (C1)-(C2) and assume without loss of generality that M and N are full column rank. To reconstruct a solution to $\mathcal{P}_\infty(\gamma)$, it suffices to find $A_K \in \mathbf{R}^{k \times k}$ for which the ARE (3.1) has a positive definite solution X_{cl} . Lemma 2.1 applied to $\mathcal{F}(G, K)(s) = C_{cl}(sI - A_{cl})^{-1}B_{cl}$ then ensures that $K(s) := C_K(sI - A_K)^{-1}B_K$ solves $\mathcal{P}_\infty(\gamma)$.

To find A_K and X_{cl} we backtrack on the steps of the necessity part. First define R, S, \bar{M}, \bar{N} from X, Y, M, N by reversing the formulas above. Using (4.4) and (C2) and observing that \bar{N} and \bar{M} are full rank, simple algebra shows existence of matrices $\theta > 0$ and $\psi > 0$ such that X_{cl} and Y_{cl} given by (A.1) satisfy $X_{cl} > 0$ and $Y_{cl}X_{cl} = \gamma^{-2}I$. Specifically, θ and ψ are the unique solutions of

$$R\bar{N} + \bar{M}\psi = 0; \quad \bar{M}^T S + \theta\bar{N}^T = 0. \quad (\text{A.4})$$

With X_{cl} thus defined, observe from the (1,2) block of (3.1) that any compatible A_K must satisfy:

$$(A + B_1 B_1^T S)^T \bar{N} + \bar{N} A_K + S B_2 C_K + (C_2^T + \bar{N} B_K) B_K^T \psi = 0.$$

Because of the first equation in (A.4) and the full rank of \bar{M} , this is equivalent to

$$\bar{N} A_K \bar{M}^T = -(A + B_1 B_1^T S)^T (\gamma^{-2}I - SR) - S B_2 C_K \bar{M}^T + (C_2^T + \bar{N} B_K) B_K^T \bar{N}^T R \quad (\text{A.5})$$

which uniquely determines A_K since \bar{M} and \bar{N} are full rank. Note that (A.5) is equivalent to (4.6) in virtue of (4.3).

So far, $X_{cl} > 0$ and $A_K \in \mathbf{R}^{k \times k}$ have been reconstructed from the data X, Y, M, N, B_K, C_K according to the formulas obtained in the necessity part. To complete the proof, we must now check that (3.1) is actually satisfied for these choices of A_K and X_{cl} . As mentioned in Section 3, (3.1) is equivalent to the three equations (3.4)-(3.6). Recalling that (3.4) is equivalent to (4.3) and that A_K was chosen to satisfy (3.5), we only need to verify (3.6). The first step consists of eliminating A_K by using (A.5). Without loss of generality since \bar{M} is right invertible, pre- and

post-multiply (3.6) by \bar{M} and \bar{M}^T , respectively, and use the first identity in (A.4) to rewrite (3.5) as:

$$-\bar{M}A_K^T\bar{N}^TR - R\bar{N}A_K\bar{M}^T + R\bar{N}B_KB_K^T\bar{N}^TR + \gamma^{-2}\bar{M}C_K^TC_K\bar{M}^T + \bar{M}C_K^TB_2^T\bar{N}\bar{M}^T + \bar{M}\bar{N}^TB_2C_K\bar{M}^T + \bar{M}\bar{N}^TB_1B_1^T\bar{N}\bar{M}^T = 0. \quad (\text{A.6})$$

In virtue of (A.5) and the identity $\bar{M}\bar{N}^T = \gamma^{-2}I - RS$, this is equivalent to:

$$(\gamma^{-2}I - RS)(AR + B_1B_1^TSR) + (RA^T + RSB_1B_1^T)(\gamma^{-2}I - SR) + (\gamma^{-2}I - RS)B_1B_1^T(\gamma^{-2}I - SR) + \gamma^{-2}\{\bar{M}C_K^TB_2^T + B_2C_K\bar{M}^T + \bar{M}C_K^TC_K\bar{M}^T\} - R\{C_2B_K^T\bar{N}^T + \bar{N}B_KC_2^T + \bar{N}B_KB_K^T\bar{N}^T\}R = 0.$$

Regroup and simplify the terms involving $B_1B_1^T$ to get:

$$(\gamma^{-2}I - RS)AR + RA^T(\gamma^{-2}I - SR) + \gamma^{-2}\{\gamma^{-2}B_1B_1^T + \bar{M}C_K^TB_2^T + B_2C_K\bar{M}^T + \bar{M}C_K^TC_K\bar{M}^T\} - R\{SB_1B_1^TS + C_2B_K^T\bar{N}^T + \bar{N}B_KC_2^T + \bar{N}B_KB_K^T\bar{N}^T\}R = 0$$

or equivalently:

$$\gamma^{-2}\{AR + RA^T + \gamma^{-2}B_1B_1^T + \bar{M}C_K^TB_2^T + B_2C_K\bar{M}^T + \bar{M}C_K^TC_K\bar{M}^T\} - R\{A^TS + SA + SB_1B_1^TS + C_2B_K^T\bar{N}^T + \bar{N}B_KC_2^T + \bar{N}B_KB_K^T\bar{N}^T\}R = 0.$$

Invoking the counterparts (A.2)-(A.3) of (4.2)-(4.3) in terms of R, S, \bar{M}, \bar{N} , this last equation reduces to

$$\gamma^{-2}\{-RC_1^TC_1R\} - R\{-\gamma^{-2}C_1^TC_1\}R = 0$$

which is trivially satisfied.

Summing up, we have constructed a controller state matrix A_K for which (3.1) has a positive definite solution X_{cl} . If (A_K, B_K, C_K) is minimal, it readily follows from the converse of Lemma 2.1 that $K(s) := C_K(sI - A_K)^{-1}B_K$ solves $\mathcal{P}_\infty(\gamma)$. More generally, it can be shown that given $X_{cl} > 0$ solving (3.1), either K solves $\mathcal{P}_\infty(\gamma)$ or there exists a nonzero vector w such that:

$$A_{cl}w = j\alpha w; \quad C_{cl}w = 0; \quad A_{cl}^T(X_{cl}w) = j\alpha(X_{cl}w); \quad B_{cl}^T(X_{cl}w) = 0.$$

In the second event, $j\alpha$ must be (A_K, B_K) uncontrollable and (C_K, A_K) unobservable since (A, B_1, C_1) was assumed minimal. Accordingly, w and $X_{cl}w$ must be of the form $\begin{pmatrix} 0 \\ \star \end{pmatrix}$ when partitioned as A_{cl} , which contradicts the full rank assumption on N . Hence K must solve $\mathcal{P}_\infty(\gamma)$. Finally, the equations (4.5)-(4.7) for A_K are easily derived by writing the (1,2) blocks of (3.1) and (3.8) and by invoking (A.4) and the GRE's (A.2)-(A.3) to simplify the resulting equations. ■

Appendix B

Proof of Lemma 5.1:

The proof requires working with the duals of (5.1)-(5.2). In terms of $R_0 := X_0^{-1} > 0$, (5.1) becomes

$$AR_0 + R_0A^T + R_0(C^TC + Q_0)R_0 + P_0 - BB^T = 0 \quad (\text{B.7})$$

and we are concerned with the existence of an *antistabilizing* solution R to

$$AR + RA^T + R(C^TC + Q) + P - BB^T = 0. \quad (\text{B.8})$$

A stabilizing solution of (5.2) is then obtained as $X(P, Q) := R^{-1}$.

A difficulty with (B.7) is the indefinite term $P_0 - BB^T$ which can be removed as follows. Introduce the Hamiltonian matrix $H_0 := \begin{pmatrix} A^T & C^T C + Q_0 \\ BB^T - P_0 & -A \end{pmatrix}$ associated with (B.7) and the antistabilizing solution π of $A\pi + \pi A^T + \pi(C^T C + Q)\pi - BB^T = 0$. The assumptions on (A, B) and $(C, -A)$ guarantee existence and nonsingularity of this solution. Now, it is easily verified that H_0 is similar to

$$H_r = \begin{pmatrix} I & 0 \\ -\pi & I \end{pmatrix} H_0 \begin{pmatrix} I & 0 \\ \pi & I \end{pmatrix} = \begin{pmatrix} A_\pi^T & C^T C + Q_0 \\ -P_0 + \pi(Q - Q_0)\pi & -A_\pi \end{pmatrix} \quad (\text{B.9})$$

where $A_\pi := A + \pi(C^T C + Q)$ is antistable. Observe that the off-diagonal terms of H_r are now semi-definite.

Since R_0 solves (B.7), $\begin{pmatrix} I \\ R_0 \end{pmatrix}$ is H_0 -invariant and consequently $\begin{pmatrix} I & 0 \\ -\pi & I \end{pmatrix} \begin{pmatrix} I \\ R_0 \end{pmatrix} = \begin{pmatrix} I \\ R_0 - \pi \end{pmatrix}$ is H_r -invariant. Hence $R_0 - \pi$ satisfies

$$A_\pi(R_0 - \pi) + (R_0 - \pi)A_\pi^T + (R_0 - \pi)(C^T C + Q_0)(R_0 - \pi) + P_0 + \pi(Q_0 - Q)\pi = 0$$

which implies that

$$A_\pi(R_0 - \pi) + (R_0 - \pi)A_\pi^T + (R_0 - \pi)(C^T C + Q)(R_0 - \pi) + P \leq 0. \quad (\text{B.10})$$

It follows from a simple extension of Lemma 2.1 that

$$\|\Lambda(sI - A_\pi)^{-1}L\|_\infty \leq 1 \quad (\text{B.11})$$

where Λ and L are any matrices satisfying $C^T C + Q = \Lambda^T \Lambda$ and $P = LL^T$.

First assume strict inequality in (B.11). Then the Hamiltonian $\begin{pmatrix} A_\pi^T & C^T C + Q \\ -P & -A_\pi \end{pmatrix}$ has no imaginary axis eigenvalue as well as $H := \begin{pmatrix} A^T & C^T C + Q \\ BB^T - P & -A \end{pmatrix}$ in virtue of the similarity (B.9) specialized to $P_0 = P$ and $Q_0 = Q$. Invoking the detectability of $(C, -A)$, we conclude to the existence of an antistabilizing solution R to (B.8). Now, subtract (5.1) from (5.2) to obtain

$$A_\pi^T(R - R_0) + (R - R_0)A_\pi = (R - R_0)(Q + C^T C P)(R - R_0) + R_0(Q - Q_0)R_0 + P - P_0.$$

Observing that the right-hand side is nonnegative and recalling that A_π is antistable, it follows that $R - R_0 \geq 0$. Hence $R \geq R_0 \geq 0$ so that $X(P, Q) := R^{-1}$ is a stabilizing solution of (5.2) and satisfies $0 < X(P, Q) \leq X_0$. The case where (B.9) is an equality is deduced by a limiting argument. Note that stabilizing must then be weakened to strong.

Finally, we establish that instability of $A + (P - BB^T)X$ only occurs in boundary cases where P cannot be uniformly increased without losing solvability. Indeed, suppose $X(P, Q)$ is not stabilizing or equivalently, that R is not antistabilizing. Then (B.11) is an equality from the previous discussion. Now if in the meantime (5.3) had a positive definite solution X_ϵ for some $\epsilon > 0$, $X_\epsilon^{-1} - \pi$ would solve (B.10) with P replaced by $(1 + \epsilon)P$. In turn, this would require $\|\Lambda(sI - A_\pi)^{-1}L\|_\infty \leq \frac{1}{\sqrt{1 + \epsilon}} < 1$, a contradiction to the fact that (B.11) is an equality. Conversely, if $X(P, Q)$ is stabilizing then continuity properties of stabilizing solutions [4] ensure that (5.3) retains a positive definite (stabilizing) solution for ϵ small enough. ■

Appendix C

Proof of Lemma 6.1:

For simplicity, assume that (A_0, B_0, C_0) is minimal and look for embeddings of F_0 of the form $\Gamma = \left(\begin{array}{c|cc} A_0 & B_0 & H \\ \hline C_0 & 0 & I \\ L & I & 0 \end{array} \right)$. Given a minimal realization $Q := \bar{C}(sI - \bar{A})^{-1}\bar{B}$, $F(Q) := \mathcal{F}(\Gamma, Q)$ is realized by $\bar{A} = \begin{pmatrix} A_0 & H\bar{C} \\ \bar{B}L & \bar{A} \end{pmatrix}$, $\bar{B} = \begin{pmatrix} B_0 \\ \bar{B} \end{pmatrix}$, $\bar{C} = (C_0, \bar{C})$. Hence, $F(Q)$ satisfies the H_∞ constraints iff. there exists $\bar{X} = \bar{X}^T > 0$ solving

$$\bar{A}^T \bar{X} + \bar{X} \bar{A} + \bar{X} \bar{B} \bar{B}^T \bar{X} + \gamma^{-2} \bar{C}^T \bar{C} = 0. \quad (C.1)$$

To fully decouple the equations relative to F_0 and Q , seek H and L so that (C.1) has a solution of the form $\bar{X} = \begin{pmatrix} X_0 & 0 \\ 0 & \bar{X} \end{pmatrix}$. This requires $(\gamma^{-2} C_0^T + X_0 H) \bar{C} + (L^T + X_0 B_0) \bar{B}^T \bar{X} = 0$ which is trivially satisfied when choosing $H := -\gamma^{-2} X_0^{-1} C_0^T$ and $L := -B_0^T X_0$. With this choice, it is easily verified that the spectrum of the Hamiltonian matrix $\begin{pmatrix} \bar{A} & \bar{B}^T \bar{B} \\ -\gamma^{-2} \bar{C}^T \bar{C} & -\bar{A}^T \end{pmatrix}$ is the union of those of $\begin{pmatrix} A_0 & B_0^T B_0 \\ -\gamma^{-2} C_0^T C_0 & -A_0^T \end{pmatrix}$ and of $\begin{pmatrix} \bar{A} & \bar{B}^T \bar{B} \\ -\gamma^{-2} \bar{C}^T \bar{C} & -\bar{A}^T \end{pmatrix}$. Consequently, $\|F(Q)\|_\infty < \gamma$ if and only if $\|Q\|_\infty < \gamma$. In addition, if $Q \in \mathcal{BR}_\infty(\gamma)$ then \bar{X} is positive definite and can be chosen as the stabilizing solution of $\bar{A}^T \bar{X} + \bar{X} \bar{A} + \bar{X} \bar{B} \bar{B}^T \bar{X} + \gamma^{-2} \bar{C}^T \bar{C} = 0$. Then \bar{X} is a positive stabilizing solution of (C.1) which guarantees internal stability of $F(Q)$. Conversely, $\bar{X} > 0$ ensures that $\bar{X} > 0$ and that $Q(s)$ is stable.

Finally, we use the characterization of Theorem 5.1 in [9] to show that Γ_{F_0} is all-pass when $\gamma = 1$. Indeed, the Lyapunov equations associated with Γ_{F_0} are then

$$\begin{aligned} A_0 P + P A_0^T + (B_0, -X_0^{-1} C_0^T)(B_0, -X_0^{-1} C_0^T)^T &= 0; \\ A_0^T Q + Q A_0 + (C_0, -B_0^T X_0)^T (C_0, -B_0^T X_0) &= 0 \end{aligned}$$

or equivalently

$$A_0 P + P A_0^T + B_0 B_0^T + X_0^{-1} C_0^T C_0 X_0^{-1} = 0; \quad A_0^T Q + Q A_0 + X_0 B_0 B_0^T X_0 + C_0^T C_0 = 0.$$

Comparing with (6.4), we obtain $P = X_0^{-1}$ and $Q = X_0$ whence $PQ = I$ and Γ_{F_0} is all-pass. ■

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